Approximation of general shell problems by flat plate elements (part 3 : extension to triangular curved facet elements)

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Abstract :

An alternative to the approximation of general thin shell problems by flat plate elements (see Part 1 and 2) is proposed : the middle surface is now approached by curved triangular facets. It consists of a P_2-Lagrange interpolation of the geometry of the shell, while the displacement field is approximated by triangles of type (1) for the membrane components and by reduced H.C.T. triangles for the bending component. To make the implementation easier, we also consider the addition of the drilling degree of freedom as a sixth independent degree of freedom by node (see Part 2). We prove the convergence of the method for arbitrary thin shells.

APPROXIMATION DE PROBLEMES DE COQUES GENERALES PAR ELEMENTS DE PLAQUES
(Partie 3 : Extension aux éléments à facettes triangulaires courbes)

Résumé :

Nous proposons ici une alternative à l'approximation de problèmes de coques minces générales par éléments de plaques (voir les parties 1 et 2 de cette étude) : la surface moyenne est maintenant approchée par des facettes triangulaires courbes. Il s'agit d'une interpolation P_2-Lagrange de la géométrie de la coque, alors que le champ de déplacement est approché par des triangles de type (1) pour les composantes de membrane et par des triangles H.C.T.-réduits pour la composante de flexion. Afin de faciliter l'implémentation, nous considérons également l'addition d'un sixième degré de liberté par noeud (voir partie 2). Nous prouvons la convergence de la méthode pour des coques minces de formes arbitraires.

1 INTRODUCTION

Among the various *pseudo-conforming* displacement finite element methods used by engineers for the analysis of general shell problems, the *doubly curved facet elements* approach may appear to be a good challenge with respect to some conforming methods (see BERNADOU (1980), CIARLET (1976), DUPUIS-GOEL (1970)). Indeed, for most of classical geometries (spherical, circular cylindrical, circular conical shells), this is actually a conforming method ; whereas, in case of more complex geometries, for which the euclidean coordinates of a finite number of points is the only data available, it should constitute an efficient alternative to the *method of flat shell elements* (see Part 1 and 2 of this work).

In this paper, we consider a P_2-Lagrange interpolation upon each triangle for the approximation of the geometry of the middle-surface. This kind of approximation is still nonconforming since KOITER's general thin shell theory involves derivatives of the third order of the mapping $\vec{\phi}$ and such terms cannot be estimated by polynomials of the second degree. Then, the tangential components of the displacements are approximated by triangles of type (1), while the transverse component is approximated by reduced H.C.T. triangles. For engineering computations, it is convenient to introduce a sixth degree of freedom at each vertex of the triangulation, i.e. the drilling degree of freedom (see BATHE (1982), ZIENKIEWICZ (1977), and Part 2 of this work), which is approximated by triangles of type (1).

By means of simple compatibility conditions, which impose the continuity of the displacement and rotation vectors at each vertex of the triangulation, we can establish *convergence* for general shells, without any restriction upon the geometry of the middle surface, nor even the definition of corrector terms.

Notations and references : In this study, we will use as constant references notations and results of BERNADOU-DUCATEL-TROUVÉ (Part 1), and BERNADOU-TROUVÉ (Part 2) ; specific formula or theorems will be respectively recorded by adding a "I", or a "II" symbol ahead of their designation.

For instance, the continuous problem and the general linear thin shell theory of KOITER are described in (I, § 2) and a reference conforming approximation of this problem is presented in (I, § 3).

CONTENTS :

2 A NONCONFORMING APPROXIMATION BY TRIANGULAR CURVED FACETS

In this paragraph, we consider an approximation of the geometry of the shell that leads to constant curvatures over each triangle. Then, we introduce the new discrete problem associated with the finite element space $\vec{\vec{X}}_h$ (already described in (I, section 4.2)).

2.1. The approximate middle surface $\overline{\mathcal{S}}_h$

Now, we consider the following *approximation of the mapping* $\vec{\phi} = \phi^i\vec{e}_i$, defined *triangle by triangle* : each component ϕ^i, i=1,2,3, is replaced by its interpolant ϕ_h^i in the finite element space Φ_h such that :

$$\left.\begin{array}{l}\text{(i) on each triangle } K \in \mathcal{T}_h\text{, the functions of } \Phi_h \text{ belongs to } P_2(K)\ ;\\ \text{(ii) on each triangle } K \in \mathcal{T}_h\text{, the functions of } \Phi_h \text{ are completely}\\ \text{determined by their values at the vertices } \Sigma_i,\ 1\leq i\leq 3\text{, and at the}\\ \text{midside points } \Sigma_{jk},\ 1\leq j<k\leq 3\text{, of } K\ ;\\ \text{(iii) } \Phi_h \subset \mathcal{C}^0(\overline{\Omega}).\end{array}\right\}\quad(2.1.1)$$

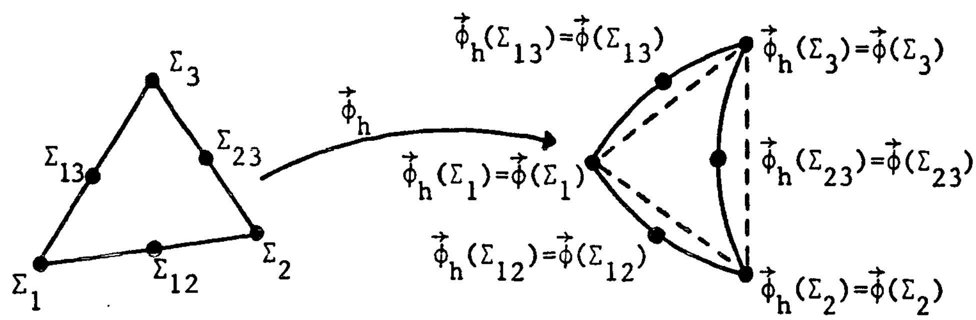

Figure 2.1.1 : A triangular curved facet

Let $\vec\phi_h \in (\Phi_h)^3$ be the interpolant of $\vec\phi$. This approximation amounts to replace the continuous middle surface $\overline{\mathscr{S}}$ by a middle surface $\overline{\mathscr{S}}_h$ with triangular *curved facets*. By construction the images by $\vec\phi_h$ of the vertices and of the midsidepoints of the triangulation $\mathscr{T}_h$ of the reference domain Ω belong to the continuous middle surface $\overline{\mathscr{S}}$.

In order to apply results already obtained in Part 1 and 2, we only consider here families of *almost-affine* finite elements – this excludes the isoparametric approach. Thus the triangulation $\mathscr{T}_h$ of the domain Ω is obtained from a reference triangle $\hat T$, and from a *regular affine* mapping F_K defined by :

$$F_K : \hat x \in \hat K \to F_K(\hat x) = B_K\hat x + b_K = x \in K ,$$

where B_K is an invertible matrix, and b_K a vector of $\mathbb{R}^2$; F_K is uniquely determined by the data : $F_K(\hat\Sigma_i) = \Sigma_i$, $1\le i\le 3$, where $\hat\Sigma_i, \Sigma_i$ respectively denote the vertices of triangles $\hat K$ and K. In particular, if $\lambda = (\lambda_1,\lambda_2,\lambda_3)$ denotes the barycentric coordinates of any point in $\hat K$, we have : $\xi = (\xi^1,\xi^2) = F_K(\lambda)$.

Then, to each curved facet $k = \vec\phi_h(K)$, $K \in \mathscr{T}_h$, we can associate a local basis $\vec a_{h\alpha} = \vec\phi_{h,\alpha}$, $\sqrt{a_h} = |\vec a_{h1}\times\vec a_{h2}|$, $\vec a_{h3} = (\vec a_{h1}\times\vec a_{h2})/\sqrt{a_h} = \vec a_h^{3}$, and fundamental forms : $a_{h\alpha\beta} = \vec a_{h\alpha}\cdot\vec a_{h\beta}$, $b_{h\alpha\beta} = \vec a_{h3}\cdot\vec a_{h\beta,\alpha}$, $b_{h\alpha}^{\beta} = a_h^{\beta\lambda} b_{h\lambda\alpha}$, $\Gamma_{h\beta\gamma}^{\alpha} = \vec a_h^{\alpha}\cdot\vec a_{h\beta,\gamma}$, etc. Each of these quantities are no longer constant on a triangle, and may present discontinuities at the interface between two facets. Let us add that the matrix $a_{h\alpha\beta}$ is invertible provided that h is sufficiently small as we have : $a_h = \det(a_{h\alpha\beta}) = a + O(h^2)$, and $\det a \neq 0$ (but it does not appear to be a significative restriction on $\mathscr{T}_h$).

2.2. The discrete problem

Let us refer to the definitions (I, (3.1.1) and (3.1.2)) for the discrete finite element spaces $\tilde{\tilde{X}}_{h1}$ and $\tilde{\tilde{X}}_{h2}$ (respectively associated to triangles of type (1), and to reduced H.C.T. triangles). Following the steps of (II, § 2), it is possible to associate to the displacement field $\vec{\tilde{\tilde{v}}}_h \in \vec{\tilde{\tilde{X}}}_h = \tilde{\tilde{X}}_{h1} \times \tilde{\tilde{X}}_{h1} \times \tilde{\tilde{X}}_{h2}$, defined on each triangular curved facet, the rotation vector $\vec{\tilde{\tilde{\omega}}}_h$, i.e.

$$\vec{\tilde{\tilde{\omega}}}_h(\vec{\tilde{\tilde{v}}}_h) = \tilde{\tilde{\omega}}_h^i \vec{a}_{hi} \;, \text{ with : } \tilde{\tilde{\omega}}_h^\lambda = \frac{1}{\sqrt{a_h}}\, e^{\lambda\mu}(\tilde{\tilde{v}}_{h3,\mu} + b_{h\mu}^{\nu}\tilde{\tilde{v}}_{h\nu}) \;, \tag{2.2.1}$$

and where the third component $\tilde{\tilde{\omega}}_h^3$ belongs to the space $\tilde{\tilde{X}}_{h1}$. Therefore, the values $\tilde{\tilde{\omega}}_h^3(\Sigma)$ ($\forall\Sigma$ vertex of $\mathcal{T}_h$) are new degrees of freedom, *independent* of $\vec{\tilde{\tilde{v}}}_h$. In particular, this *drilling* degree of freedom cannot be understood as an approximation of the rotation around the normal to the facet. The introduction of this sixth degree of freedom at the vertices of $\mathcal{T}_h$ allows us to define *compatibility conditions*, easy to implement, i.e.

C1) *the displacement* $\vec{\tilde{\tilde{v}}}_h$ *is continuous at the vertices* Σ *of the triangulation* $\mathcal{T}_h$:

$$\vec{\tilde{\tilde{v}}}_h(\Sigma^+) = \vec{\tilde{\tilde{v}}}_h(\Sigma^-) \quad , \quad \forall\Sigma^+ = \Sigma^- = \Sigma \text{ vertex of } \mathcal{T}_h \; (\Sigma^+ \in K^+,\ \Sigma^- \in K^-) \;; \tag{2.2.2}$$

C2) *the vector* $\vec{\tilde{\tilde{\omega}}}_h$ *is continuous at the vertices* Σ *of the triangulation* $\mathcal{T}_h$;

$$\vec{\tilde{\tilde{\omega}}}_h(\Sigma^+) = \vec{\tilde{\tilde{\omega}}}_h(\Sigma^-) \quad , \quad \forall\Sigma^+ = \Sigma^- = \Sigma \text{ vertex of } \mathcal{T}_h \; (\Sigma^+ \in K^+,\ \Sigma^- \in K^-). \tag{2.2.3}$$

The relations (2.2.2) (2.2.3) enable to define the discrete space $\vec{\tilde{Y}}_h$

$$\vec{\tilde{Y}}_h = \{(\vec{\tilde{\tilde{v}}}_h, \tilde{\tilde{\omega}}_h^3) \in \vec{\tilde{\tilde{X}}}_h \times \tilde{\tilde{X}}_{h1} \;;\; (\vec{\tilde{\tilde{v}}}_h, \tilde{\tilde{\omega}}_h^3) \text{ satisfies the compatibility conditions (2.2.2) and (2.2.3)}\} \;,$$

(with dim $\vec{\tilde{Y}}_h = 6N_h$, N_h being the number of vertices of $\mathcal{T}_h$), and its subspace $\vec{\tilde{W}}_h$, which takes into account the boundary conditions, i.e.

$$\vec{\tilde{W}}_h = \{(\vec{\tilde{v}}_h, \tilde{\omega}_h^3) \in \vec{\tilde{Y}}_h \quad , \quad \vec{\tilde{v}}_h(\Sigma) = \vec{0} \text{ and } \vec{\tilde{\omega}}_h(\Sigma) = \vec{0} \quad , \quad \forall\Sigma \in \Gamma_o\} \;; \tag{2.2.4}$$

we equip these discrete spaces with the norm :

$$(\vec{\tilde{v}}_h, \tilde{\omega}_h^3) \in \vec{\tilde{Y}}_h \to \sum_{K\in\mathcal{T}_h} [\|\tilde{v}_{h1}\|_{1,K}^2 + \|\tilde{v}_{h2}\|_{1,K}^2 + \|\tilde{v}_{h3}\|_{2,K}^2 + \|\tilde{\omega}_{h3}\|_{0,K}^2]^{1/2} \;. \tag{2.2.5}$$

Next, we prove the *existence* of a bijection between the spaces $\vec{\tilde{W}}_h$ and

$$\vec{W}_h = \{(\vec{v}_h, \omega_h^3) \in \vec{X}_h \times X_{h1} \;;\; \vec{v}_{h|\Gamma_o} = \vec{0} \text{ and } \vec{\omega}_{h|\Gamma_o} = \vec{0}\} \subset (H^1(\Omega))^2 \times H^2(\Omega) \times H^1(\Omega) \;,$$

(see (II, remark 2.2.1) which concerns the introduction of a sixth boundary condition, i.e., $\omega^3_{h|\Gamma_o} = 0$).

Theorem 2.2.1 : The compatibility relations

$$\vec{\vec{v}}_h(\Sigma) = \vec{v}_h(\Sigma) \quad \textit{and} \quad \vec{\vec{\omega}}_h(\Sigma) = \vec{\omega}_h(\Sigma) \ , \quad \forall\Sigma \ \textit{vertex of} \ \mathcal{T}_h \ , \tag{2.2.6}$$

define a bijection $F_h : \vec{\vec{W}}_h \to \vec{W}_h$, *that associates to each function* $(\vec{\vec{v}}_h, \tilde{\omega}^3_h) \in \vec{\vec{W}}_h$ *one and only one function* $(\vec{v}_h, \omega^3_h) \in \vec{Y}_h$.

Proof : This proof is similar to the proof of (II, Theorem 2.2.1). On the one hand, relations (II, (2.2.5)) are still available ; on the other hand, we derive from (2.2.6) :

$$\tilde{v}_{hi}(\Sigma) = v_{hj}(\Sigma)(\vec{a}^j(\Sigma).\vec{a}_{hi}(\Sigma)) \ , \ 1 \le i \le 3 \ , \tag{2.2.7}$$

$$\left.\begin{aligned} \tilde{v}_{h3,\mu}(\Sigma) = &\sqrt{\frac{a_h(\Sigma)}{a(\Sigma)}}\, e_{\lambda\mu} e^{\kappa\nu} [v_{h3,\nu}(\Sigma) + b^{\epsilon}_{\nu}(\Sigma) v_{h\epsilon}(\Sigma)] (\vec{a}_{\kappa}(\Sigma).\vec{a}^{\lambda}_h(\Sigma)) + \\ &+ \sqrt{a_h(\Sigma)}\, e_{\lambda\mu} \omega^3_h(\Sigma)(\vec{a}_3(\Sigma).\vec{a}^{\lambda}_h(\Sigma)) - b^{\nu}_{h\mu}(\Sigma) v_{hj}(\Sigma)(\vec{a}^j(\Sigma).\vec{a}_{h\nu}(\Sigma)) \ , \end{aligned}\right\} \tag{2.2.8}$$

$$\left.\begin{aligned} \tilde{\omega}^3_h(\Sigma) = &\frac{1}{\sqrt{a(\Sigma)}}\, e^{\kappa\nu} [v_{h3,\nu}(\Sigma) + b^{\epsilon}_{\nu}(\Sigma) v_{h\epsilon}(\Sigma)] (\vec{a}_{\kappa}(\Sigma).\vec{a}^3_h(\Sigma)) \\ &+ \omega^3_h(\Sigma)(\vec{a}_3(\Sigma).\vec{a}^3_h(\Sigma)) \end{aligned}\right\} \tag{2.2.9}$$

where Σ denotes any of the three vertices of a given triangle K of $\mathcal{T}_h$. So, we conclude in the same way as in (II, Theorem 2.2.1). Finally, there is no difficulty to verify that the compatibility relations (2.2.6) preserve the boundary conditions.

□

Now, to each displacement field $(\vec{\vec{v}}_h, \tilde{\omega}^3_h) \in \vec{\vec{Y}}_h$ of the faceted surface $\mathcal{S}_h$, we can associate a strain tensor $(\tilde{\gamma}_{h\alpha\beta})$ and a change of curvature tensor $(\tilde{\rho}_{h\alpha\beta})$, which are defined, triangle by triangle, as follows :

$$\tilde{\gamma}_{h\alpha\beta}(\vec{\vec{v}}_h) = [\tfrac{1}{2}(\tilde{v}_{h\beta,\alpha} + \tilde{v}_{h\alpha,\beta}) - \Gamma^{\lambda}_{h\alpha\beta}\tilde{v}_{h\lambda} - b_{h\alpha\beta}\tilde{v}_{h3}] \ , \tag{2.2.10}$$

$$\left.\begin{aligned} \tilde{\rho}_{h\alpha\beta}(\vec{\vec{v}}_h) = &\tilde{v}_{h3,\alpha\beta} - \Gamma^{\lambda}_{h\alpha\beta}\tilde{v}_{h\lambda} - b^{\lambda}_{h\alpha} b_{h\lambda\beta}\tilde{v}_{h3} + b^{\lambda}_{h\alpha}\tilde{v}_{h\lambda,\beta} + b^{\lambda}_{h\beta}\tilde{v}_{h\lambda,\alpha} + \\ &+ (b^{\lambda}_{h\beta,\alpha} - \Gamma^{\lambda}_{h\mu\beta} b^{\mu}_{h\alpha} - \Gamma^{\mu}_{h\alpha\beta} b^{\lambda}_{h\mu})\tilde{v}_{h\lambda} \ . \end{aligned}\right\} \tag{2.2.11}$$

Then, the variational formulation of the *discrete problem associated to the approximate surface* $\mathscr{S}_h$ is given by :

Problem 2.2.1 : Find $(\vec{\vec{u}}_h, \tilde{\eta}_h^3) \in \vec{\vec{W}}_h$, *such that*

$$\sum_{K\in\mathscr{T}_h} \tilde{A}_{Kh}[(\vec{\vec{u}}_h,\tilde{\eta}_h^3),(\vec{\vec{v}}_h,\tilde{\omega}_h^3)] = \sum_{K\in\mathscr{T}_h} \tilde{f}_{Kh}(\vec{\vec{v}}_h) \quad , \quad \forall(\vec{\vec{v}}_h,\tilde{\omega}_h^3) \in \vec{\vec{W}}_h \ ,$$

where we have denoted :

$$\tilde{A}_{Kh}[(\vec{\vec{u}}_h,\tilde{\eta}_h^3),(\vec{\vec{v}}_h,\tilde{\omega}_h^3)] = \tilde{a}_{Kh}(\vec{\vec{u}}_h,\vec{\vec{v}}_h) + k(\tilde{\eta}_h^3,\tilde{\omega}_h^3)_{L^2(K)} \ ,$$

$$\tilde{a}_{Kh}(\vec{\vec{u}}_h,\vec{\vec{v}}_h) = \int_K \frac{Ee}{1-\nu^2} \{(1-\nu)\tilde{\gamma}_{h\beta}^{\alpha}(\vec{\vec{u}}_h)\tilde{\gamma}_{h\alpha}^{\beta}(\vec{\vec{v}}_h) + \tilde{\gamma}_{h\alpha}^{\alpha}(\vec{\vec{u}}_h)\tilde{\gamma}_{h\beta}^{\beta}(\vec{\vec{v}}_h) + \\ + \frac{e^2}{12}[(1-\nu)\ \tilde{\rho}_{h\beta}^{\alpha}(\vec{\vec{u}}_h)\tilde{\rho}_{h\alpha}^{\beta}(\vec{\vec{v}}_h) + \tilde{\rho}_{h\alpha}^{\alpha}(\vec{\vec{u}}_h)\tilde{\rho}_{h\beta}^{\beta}(\vec{\vec{v}}_h)]\} \sqrt{a_h}\ d\xi^1 d\xi^2 \ ,$$

$k > 0$ *constant, independent of* K ,

$$(\tilde{\eta}_h^3,\tilde{\omega}_h^3)_{L^2(K)} = \int_K \tilde{\eta}_h^3\tilde{\omega}_h^3 \sqrt{a_h}\ d\xi^1 d\xi^2 \ ,$$

$$\tilde{f}_{Kh}(\vec{\vec{v}}_h) = \int_K \vec{p}.\vec{\vec{v}}_h \sqrt{a_h}\ d\xi^1 d\xi^2 \ .$$

□

This amounts to approach the energy of the shell by a sum of elementary curved facet energies.

For the study of the convergence of such a method, it is convenient to associate to Problem 2.2.1 its *equivalent discrete problem formulated on the continuous middle surface* $\mathscr{S}$. By using the bijection F_h defined in Theorem 2.2.1, we obtain :

Problem 2.2.2 : Find $(\vec{u}_h^*, \eta_h^3) \in \vec{W}_h$ *such that*

$$A_h[(\vec{u}_h^*,\eta_h^3),(\vec{v}_h,\omega_h^3)] = G_h[(\vec{v}_h,\omega_h^3)] \quad , \quad \forall(\vec{v}_h,\omega_h^3) \in \vec{W}_h \ ,$$

where we have denoted :

$$A_h[(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3)] = \sum_{K\in\mathscr{T}_h} \{a_{Kh}[(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3)] + kb_{Kh}[(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3)]\} \ ,$$

$$G_h[(\vec{v}_h,\omega_h^3)] = \sum_{K\in\mathscr{T}_h} f_{Kh}(\vec{v}_h,\omega_h^3) \ ,$$

with the following correspondences for any $(\vec{v}_h,\omega_h^3) = F_h(\vec{\vec{v}}_h,\tilde{\omega}_h^3)$, $(\vec{w}_h,\tilde{\theta}_h^3) = F_h(\vec{\vec{w}}_h,\tilde{\theta}_h^3)$:

$$a_{Kh}[(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3)] = \tilde{a}_{Kh}(\vec{\vec{v}}_h,\vec{\vec{w}}_h) \ ; \ b_{Kh}[(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3)] = (\tilde{\omega}_h^3,\tilde{\eta}_h^3)_{L^2(K)} \ ;$$

$$f_{Kh}(\vec{v}_h,\omega_h^3) = \tilde{f}_{Kh}(\vec{\vec{v}}_h) \ .$$

□

3 CONVERGENCE AND ERROR ESTIMATES

In order to prove the *existence and uniqueness* of a solution $(\vec{\vec{u}}_h,\tilde{\eta}_h^3)$ for the discrete Problem 2.2.1, and the *convergence* of the method in the following sense : $(\|\vec{u}-\vec{u}_h^*\|_{\vec{V}}^2+\|\eta_h^3\|_{0,\Omega}^2)^{1/2} \le Ch\|\vec{u}\|_{\vec{V}}$, we study the *consistency* error estimates, since it was shown in (I, §5) and (II, §3) that we need such estimates to evaluate the *discretization* error $\|\vec{u}-\vec{u}_h^*\|_{\vec{V}}$.

3.1. Estimate of $|a(\vec{v}_h,\vec{w}_h) + k(\omega_h^3,\theta_h^3)_{L^2(\Omega)} - A_h[(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3)]|$

By using similar arguments as in (II, § 3.2), it amounts to evaluate

$$|\gamma_\beta^\alpha(\vec{v}_h)-\tilde{\gamma}_{h\beta}^{\alpha}(\vec{\vec{v}}_h)|_{0,K} \ , \ |\rho_\beta^\alpha(\vec{v}_h)-\tilde{\rho}_{h\beta}^{\alpha}(\vec{\vec{v}}_h)|_{0,K} \ , \ |\omega_h^3-\tilde{\omega}_h^3|_{0,K} \ , \ |\sqrt{a}-\sqrt{a_h}|_{0,\infty,K} \ .$$

Firstly, we obtain :

Theorem 3.1.1 : There exists a constant C, *independent of* h, *such that for any* $(\vec{\vec{v}}_h,\tilde{\omega}_h^3) \in \vec{\vec{W}}_h$ *and* $(\vec{v}_h,\omega_h^3) \in \vec{W}_h$ *in correspondence through the bijection* F_h, *we have :*

$$|\gamma_\beta^\alpha(\vec{v}_h)-\tilde{\gamma}_{h\beta}^{\alpha}(\vec{\vec{v}}_h)|_{0,K} \le Ch(\|v_{h1}\|_{0,K}^2 + \|v_{h2}\|_{0,K}^2 + \|v_{h3}\|_{1,K}^2 + \|\omega_h^3\|_{0,K}^2)^{1/2} \ . \quad (3.1.1)$$

Proof (in six steps)

Let us recall the definitions :

$$\gamma_\beta^\alpha(\vec{v}_h) = a^{\alpha\lambda}[\tfrac{1}{2}\ (v_{h\beta,\lambda}+v_{h\lambda,\beta}) - \Gamma_{\lambda\beta}^{\nu}v_{h\nu} - b_{\lambda\beta}v_{h3}] \ ,$$

$$\tilde{\gamma}_{h\beta}^{\alpha}(\vec{\vec{v}}_h) = a_h^{\alpha\lambda}[\tfrac{1}{2}\ (\tilde{v}_{h\beta,\lambda}+\tilde{v}_{h\lambda,\beta}) - \Gamma_{h\lambda\beta}^{\nu}\tilde{v}_{h\nu} - b_{h\lambda\beta}\tilde{v}_{h3}] \ .$$

Step 1 : Expression of $\tilde{v}_{h\beta,\lambda}$ *and* $\tilde{v}_{h\nu}$ *as functions of degrees of freedom of* $\vec{W}_h$

From the definition of the space $\tilde{\tilde{X}}_{h1}$, and by virtue of (2.2.7), we get

$$\left.\begin{aligned} \tilde{v}_{h\nu}(\xi) &= \sum_{i=1}^{3} \lambda_i \tilde{v}_{h\nu}(\Sigma_i) = \sum_{i=1}^{3} \lambda_i d_{h\nu}^{j}(\Sigma_i) v_{hj}(\Sigma_i) \ , \\ \tilde{v}_{h\beta,\lambda}(\xi) &= \sum_{i=1}^{3} \frac{\partial\lambda_i}{\partial\xi^\lambda}\,\tilde{v}_{h\beta}(\Sigma_i) = \sum_{i=1}^{3} \frac{\partial\lambda_i}{\partial\xi^\lambda}\, d_{h\beta}^{j}(\Sigma_i) v_{hj}(\Sigma_i) \ , \end{aligned}\right\} \quad (3.1.2)$$

where we have denoted : $d^j_{hk}(\xi) = \vec{a}^j(\xi).\vec{a}_{hk}(\xi)$, $\forall \xi \in K$.

Step 2 : Expression of $\tilde{v}_{h3}$ as function of degrees of freedom of $\vec{W}_h$

By definition of the space $\tilde{\tilde{X}}_{h2}$, we have on each subtriangle K_j, $1 \le j \le 3$, of K :

$$\tilde{v}_{h3}(\xi)|_{K_j} = \sum_{i=1}^{3} p^o_{j,i}(\lambda)\tilde{v}_{h3}(\Sigma_i) + \\ + \sum_{i=1}^{3} \{(\xi^\nu_{i+1}-\xi^\nu_i)p^1_{j,i,i+1}(\lambda) + (\xi^\nu_{i-1}-\xi^\nu_i)p^1_{j,i,i-1}(\lambda)\}\tilde{v}_{h3,\nu}(\Sigma_i)$$

where $p^o_{j,i}$, $p^1_{j,i,i+1}$, $p^1_{j,i,i-1}$, $1 \le i,j \le 3$, are the basis functions of the reduced HSIEH-CLOUGH-TOCHER finite element attached to the subtriangle K_j. And, by using (2.2.7) and (2.2.8), we get :

$$\left.\begin{aligned} \tilde{v}_{h3}(\xi)|_{K_j} = &\sum_{i=1}^{3} p^o_{j,i}(\lambda)d^j_{h3}(\Sigma_i)v_{hj}(\Sigma_i) + \\ &+ \sum_{i=1}^{3} \{(\xi^\nu_{i+1}-\xi^\nu_i)p^1_{j,i,i+1}(\lambda) + (\xi^\nu_{i-1}-\xi^\nu_i)p^1_{j,i,i-1}(\lambda)\} \times \\ &\quad \times \{\sqrt{\frac{a_h(\Sigma_i)}{a(\Sigma_i)}}\, e_{\lambda\nu}e^{\kappa\mu}[v_{h3,\mu}(\Sigma_i) + b^\epsilon_\mu(\Sigma_i)v_{h\epsilon}(\Sigma_i)]d^{h\lambda}_\kappa(\Sigma_i) + \\ &+ \sqrt{a_h(\Sigma_i)}\; e_{\lambda\nu}\omega^3_h(\Sigma_i)d^{h\lambda}_3(\Sigma_i) - b^\mu_{h\nu}(\Sigma_i)v_{hj}(\Sigma_i)d^j_{h\mu}(\Sigma_i)\} \, , \end{aligned}\right\} \quad (3.1.3)$$

where we have denoted : $d^{hk}_j(\xi) = \vec{a}_j(\xi).\vec{a}^k_h(\xi)$, $\forall \xi \in K$, and $d^k_{hj}(\xi) = \vec{a}^k(\xi).\vec{a}_{hj}(\xi)$, $\forall \xi \in K$.

Step 3 : Some geometrical approximations

According to CIARLET (1978, §3.1), we obtain for the definition (2.1.1)

$$|\phi^i - \phi^i_h|_{m,\infty,K} \le Ch^{3-m} |\phi^i|_{3,\infty,K} \, , \quad 0 \le m \le 3 \, , \quad 1 \le i \le 3 \, ,$$

so, we derive from the assumption $\vec{\phi} \in (\mathscr{C}^3(\bar{\Omega}))^3$:

$$\left.\begin{aligned} &\vec{a}_{h\alpha}(\xi) = \vec{a}_\alpha(\xi) + 0(h^2)\vec{c}_\alpha \, , \quad a_{h\alpha\beta}(\xi) = a_{\alpha\beta}(\xi) + 0(h^2)c_{\alpha\beta} \, , \\ &\sqrt{a_h(\xi)} = \sqrt{a(\xi)} + 0(h^2)c \, , \quad \vec{a}_{h3}(\xi) = \vec{a}_3(\xi) + 0(h^2)\vec{c}_3 \, , \\ &\qquad\qquad \vec{a}^j_h(\xi) = \vec{a}^j(\xi) + 0(h^2)\vec{c}^j \, , \\ &\Gamma^\alpha_{h\beta\gamma}(\xi) = \Gamma^\alpha_{\beta\gamma}(\xi) + 0(h)c^\alpha_{\beta\gamma} \, , \quad b_{h\alpha\beta}(\xi) = b_{\alpha\beta}(\xi) + 0(h)c_{\alpha\beta} \, , \text{ etc,} \\ &d^j_{hk}(\xi) = \delta^j_k + 0(h^2)c^j_k \, , \quad d^{hk}_j = \delta^k_j + 0(h^2)c^k_j \, . \end{aligned}\right\} \quad (3.1.4)$$

In (3.1.4), and subsequently, the different constants are independent of h, and can change from an occurence to the other.

Step 4 : Finite expansions of $\tilde{v}_{h\nu}(\xi)$ *and* $\tilde{v}_{h\beta,\lambda}(\xi)$

As we have : $v_{h\lambda} \in P_1(K)$, and $v_{h3|K_j} \in P_3(K_j)$, $1 \le j \le 3$, we get :

$$\left.\begin{aligned} v_{h\lambda}(\Sigma_i) &= v_{h\lambda}(\xi) + (\xi_i^\epsilon - \xi^\epsilon) v_{h\lambda,\epsilon} , \\ v_{h3}(\Sigma_i) &= v_{h3}(\xi) + (\xi_i^\epsilon - \xi^\epsilon) v_{h3,\epsilon}(\bar{\xi}_i) , \ \bar{\xi}_i \in [\Sigma_i, \xi] , \end{aligned}\right\} \tag{3.1.5}$$

and thus, by using (3.1.4) and the following relations :

$$\sum_{i=1}^{3} \lambda_i = 1 , \quad \sum_{i=1}^{3} \lambda_i (\xi_i^\epsilon - \xi^\epsilon) = 0 , \quad \sum_{i=1}^{3} \frac{\partial \lambda_i}{\partial \xi^\lambda} (\xi_i^\epsilon - \xi^\epsilon) = \delta_\lambda^\epsilon , \quad \frac{\partial \lambda_i}{\partial \xi^\lambda} = 0(h^{-1}) , \tag{3.1.6}$$

we obtain in (3.1.2) :

$$\left.\begin{aligned} \tilde{v}_{h\nu}(\xi) &= v_{h\nu}(\xi) + 0(h^2) \sum_{i=1}^{3} (c_\nu^j v_{hj}(\Sigma_i)) , \\ \tilde{v}_{h\beta,\lambda}(\xi) &= v_{h\beta,\lambda}(\xi) + 0(h) \sum_{i=1}^{3} (c_{\beta\lambda}^j v_{hj}(\Sigma_i)) . \end{aligned}\right\} \tag{3.1.7}$$

Step 5 : Finite expansion of $\tilde{v}_{h3}(\xi)$

By substituting (3.1.4) (3.1.5) (3.1.6) into (3.1.3), we derive :

$$\begin{aligned} \tilde{v}_{h3}(\xi)|_{K_j} = & \sum_{i=1}^{3} p_{j,i}^{o}(\lambda) v_{h3}(\Sigma_i) + \\ & + \sum_{i=1}^{3} ((\xi_{i+1}^\nu - \xi_i^\nu) p_{j,i,i+1}^1(\lambda) + (\xi_{i-1}^\nu - \xi_i^\nu) p_{j,i,i-1}^1(\lambda)) v_{h3,\nu}(\Sigma_i) + \\ & + \sum_{i=1}^{3} ((\xi_{i+1}^\nu - \xi_i^\nu) p_{j,i,i+1}^1(\lambda) + (\xi_{i-1}^\nu - \xi_i^\nu) p_{j,i,i-1}^1(\lambda)) \times \\ & \qquad \times (b_\nu^\epsilon(\Sigma_i) v_{h\epsilon}(\Sigma_i) - b_{h\nu}^\mu(\Sigma_i) v_{h\mu}(\Sigma_i)) + \\ & + 0(h^2) \sum_{i=1}^{3} (c_j^k v_{hk}(\Sigma_i) + h[c_j^\epsilon v_{h\epsilon}(\Sigma_i) + c_j^{3\mu} v_{h3,\mu}(\Sigma_i) + c_j \omega_h^3(\Sigma_i)]) , \end{aligned}$$

and, thus :

$$\tilde{v}_{h3}(\xi)|_{K_j} = v_{h3}(\xi)|_{K_j} + 0(h^2) \sum_{i=1}^{3} (c_j^k v_{hk}(\Sigma_i) + c_j^{3\mu} v_{h3,\mu}(\Sigma_i) + c_j \omega_h^3(\Sigma_i)) . \tag{3.1.8}$$

Step 6 : Obtention of the estimate (3.1.1)

Finally, by substituting (3.1.4) (3.1.7) (3.1.8) into the expression of $\tilde{\gamma}_{h\beta}^{\alpha}$, we find for any $\xi \in K$:

$$\tilde{\gamma}^{\alpha}_{h\beta}(\vec{\vec{v}}_h)|_{K_j} = \gamma^{\alpha}_{\beta}(\vec{v}_h)|_{K_j} + O(h)\sum_{i=1}^{3}(c^k_j v_{hk}(\Sigma_i) + c^{3\mu}_j v_{h3,\mu}(\Sigma_i) + c_j\omega^3_h(\Sigma_j)) .$$

Therefore, there exists a constant c, independent of h, such that we have for any $K \in \mathscr{C}_h$ (with meas(K) = $O(h^2)$)

$$|\tilde{\gamma}^{\alpha}_{h\beta}(\vec{\vec{v}}_h) - \gamma^{\beta}_{\alpha}(\vec{v}_h)|_{0,K} \le Ch^2(\sum_{k=1}^{3}|v_{hk}|^2_{0,\infty,K} + |v_{h3}|^2_{1,\infty,K} + |\omega^3_h|^2_{0,\infty,K})^{1/2} ,$$

and thus, the results of interpolation theory in Sobolev spaces (see CIARLET (1978, Theorem 3.1.2)) lead to the estimate (3.1.1).

□

Secondly, we establish the *main result of this study* (to be compared with (II, Theorem 3.2.2)) :

Theorem 3.1.2 : There exists a constant C, *independent of* h, *such that for any* $(\vec{\vec{v}}_h, \tilde{\omega}^3_h) \in \vec{\vec{W}}_h$ *and* $(\vec{v}_h, \omega^3_h) \in \vec{W}_h$ *in correspondence through the bijection* F_h, *we have :*

$$|\bar{\rho}^{\alpha}_{\beta}(\vec{v}_h) - \tilde{\rho}^{\alpha}_{h\beta}(\vec{\vec{v}}_h)|_{0,K} \le Ch(\|v_{h1}\|^2_{1,K} + \|v_{h2}\|^2_{1,K} + \|v_{h3}\|^2_{2,K} + \|\omega^3_h\|^2_{0,K})^{1/2} \quad (3.1.9)$$

Proof (in four steps)

Let us recall the expressions :

$$\begin{aligned} \bar{\rho}^{\alpha}_{\beta}(\vec{v}_h) &= a^{\alpha\nu}[v_{h3,\nu\beta} - \Gamma^{\lambda}_{\nu\beta}v_{h\lambda} - b^{\lambda}_{\nu}b_{\lambda\beta}v_{h3} + b^{\lambda}_{\beta}v_{h\lambda,\nu} + b^{\lambda}_{\nu}v_{h\lambda,\beta} + \\ &\quad + (b^{\lambda}_{\beta,\nu} - \Gamma^{\lambda}_{\mu\beta}b^{\mu}_{\nu} - \Gamma^{\mu}_{\nu\beta}b^{\lambda}_{\mu})v_{h\lambda}] , \\ \tilde{\rho}^{\alpha}_{h\beta}(\vec{\vec{v}}_h) &= a^{\alpha\nu}_h[\tilde{v}_{h3,\nu\beta} - \Gamma^{\lambda}_{h\nu\beta}\tilde{v}_{h\lambda} - b^{\lambda}_{h\nu}b_{h\lambda\beta}\tilde{v}_{h3} + b^{\lambda}_{h\beta}\tilde{v}_{h\lambda,\nu} + b^{\lambda}_{h\nu}\tilde{v}_{h\lambda,\beta} + \\ &\quad + (b^{\lambda}_{h\beta,\nu} - \Gamma^{\lambda}_{h\mu\beta}b^{\mu}_{h\nu} - \Gamma^{\mu}_{h\nu\beta}b^{\lambda}_{h\mu})\tilde{v}_{h\lambda}] . \end{aligned} \quad (3.1.10)$$

Step 1 : Some preliminary results on the geometrical approximation

Using similar arguments as in (II, Theorem 4.2.2), we need finite expansions up to the third order for the expression of $\tilde{v}_{h3}(\Sigma_i)$, up to the second order for $\tilde{v}_{h3,\nu}(\Sigma_i)$, and up to the first order for the geometrical terms in (3.1.10). The constants appearing in the following finite expansions are independent of h.

By virtue of the definition (2.1.1), we have for any $\xi \in K$:

$$\vec{\phi}_h(\xi) = \sum_{i=1}^{3}\lambda_i(2\lambda_i - 1)\vec{\phi}(\Sigma_i) + \sum_{\substack{i,j=1\\ i<j}}^{3} 4\lambda_i\lambda_j\vec{\phi}(\Sigma_{ij})$$

and thus :

$$\vec{a}_{h\alpha}(\xi) - \vec{\phi}_{h,\alpha}(\xi) = \sum_{i=1}^{3} \frac{\partial\lambda_i}{\partial\xi^\alpha} (4\lambda_i-1)\vec{\phi}(\Sigma_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{3} 4\frac{\partial\lambda_i}{\partial\xi^\alpha} \lambda_j\vec{\phi}(\Sigma_{ij}) .$$

Moreover, by using the fact that $\Sigma_{ij} = \frac{1}{2}(\Sigma_i+\Sigma_j)$, $1\leq i<j\leq 3$, we can write :

$$\vec{\phi}(\Sigma_i) = \vec{\phi}(\xi) + (\xi_i^\epsilon-\xi^\epsilon)\vec{\phi}_{,\epsilon}(\xi) + \frac{1}{2}(\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta)\vec{\phi}_{,\epsilon\eta}(\xi) +$$
$$+ \frac{1}{6}(\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta)(\xi_i^\lambda-\xi^\lambda)\vec{\phi}_{,\epsilon\eta\lambda}(\xi) + 0(h^4)\vec{c}_i ,$$

$$\vec{\phi}(\Sigma_{ij}) = \vec{\phi}(\xi) + \frac{1}{2}[(\xi_i^\epsilon-\xi^\epsilon) + (\xi_j^\epsilon-\xi^\epsilon)]\vec{\phi}_{,\epsilon}(\xi) +$$
$$+ \frac{1}{8}[(\xi_i^\epsilon-\xi^\epsilon) + (\xi_j^\epsilon-\xi^\epsilon)][(\xi_i^\eta-\xi^\eta) + (\xi_j^\eta-\xi^\eta)]\vec{\phi}_{,\epsilon\eta}(\xi) +$$
$$+ \frac{1}{48}[(\xi_i^\epsilon-\xi^\epsilon) + (\xi_j^\epsilon-\xi^\epsilon)][(\xi_i^\eta-\xi^\eta) + (\xi_j^\eta-\xi^\eta)][(\xi_i^\lambda-\xi^\lambda) + (\xi_j^\lambda-\xi^\lambda)]\vec{\phi}_{,\epsilon\eta\lambda}(\xi) +$$
$$+ 0(h^4)\vec{c}_{ij} .$$

Next, by noticing that we get from relations (3.1.6) :

$$\sum_{i=1}^{3} [\frac{\partial\lambda_i}{\partial\xi^\alpha}(4\lambda_i-1) + \sum_{\substack{j=1\\ j\neq i}}^{3} 4\frac{\partial\lambda_i}{\partial\xi^\alpha}\lambda_j] = 3\sum_{i=1}^{3}\frac{\partial\lambda_i}{\partial\xi^\alpha} = 0 ,$$

$$\sum_{i=1}^{3} \{\frac{\partial\lambda_i}{\partial\xi^\alpha}(4\lambda_i-1)(\xi_i^\epsilon-\xi^\epsilon) + \sum_{\substack{j=1\\ j\neq i}}^{3} \frac{\partial\lambda_i}{\partial\xi^\alpha}\lambda_j[\frac{1}{2}(\xi_i^\epsilon-\xi^\epsilon) + \frac{1}{2}(\xi_j^\epsilon-\xi^\epsilon)]\} =$$
$$= \sum_{i=1}^{3}\frac{\partial\lambda_i}{\partial\xi^\alpha}(\xi_i^\epsilon-\xi^\epsilon) = \delta_\alpha^\epsilon ,$$

$$\sum_{i=1}^{3}\frac{\partial\lambda_i}{\partial\xi^\alpha}\{\frac{1}{2}(4\lambda_i-1)(\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta) +$$
$$+ \sum_{\substack{j=1\\ j\neq i}}^{3}\frac{1}{2}\lambda_j[(\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta) + (\xi_i^\epsilon-\xi^\epsilon)(\xi_j^\eta-\xi^\eta) + (\xi_j^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta) + (\xi_j^\epsilon-\xi^\epsilon)(\xi_j^\eta-\xi^\eta)]\} =$$
$$= \sum_{i=1}^{3}\frac{1}{2}\frac{\partial\lambda_i}{\partial\xi^\alpha}\{-(\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta) +$$
$$+ \sum_{j=1}^{3}\lambda_j[(\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta) + (\xi_i^\epsilon-\xi^\epsilon)(\xi_j^\eta-\xi^\eta) + (\xi_j^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta) + (\xi_j^\epsilon-\xi^\epsilon)(\xi_j^\eta-\xi^\eta)]\} = 0,$$

$$\sum_{i=1}^{3} \frac{\partial\lambda_i}{\partial\xi^\alpha} \{\frac{1}{6} (4\lambda_i-1)(\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta)(\xi_i^\lambda-\xi^\lambda) +$$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{3} \frac{1}{12} \lambda_j [(\xi_i^\epsilon-\xi^\epsilon) + (\xi_j^\epsilon-\xi^\epsilon)] [(\xi_i^\eta-\xi^\eta) + (\xi_j^\eta-\xi^\eta)] [(\xi_i^\lambda-\xi^\lambda) + (\xi_j^\lambda-\xi^\lambda)]\} =$$

$$= \sum_{i=1}^{3} \frac{\partial\lambda_i}{\partial\xi^\alpha} \{- \frac{1}{6} (\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta)(\xi_i^\lambda-\xi^\lambda) +$$

$$+ \sum_{j=1}^{3} \frac{1}{12} \lambda_j [(\xi_i^\epsilon-\xi^\epsilon) + (\xi_j^\epsilon-\xi^\epsilon)] [(\xi_i^\eta-\xi^\eta) + (\xi_j^\eta-\xi^\eta)] [(\xi_i^\lambda-\xi^\lambda) + (\xi_j^\lambda-\xi^\lambda)]\} =$$

$$= \sum_{i=1}^{3} \{- \frac{1}{12} \frac{\partial\lambda_i}{\partial\xi^\alpha} (\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta)(\xi_i^\lambda-\xi^\lambda) +$$

$$+ \frac{1}{12} \lambda_i [(\xi_i^\eta-\xi^\eta)(\xi_i^\lambda-\xi^\lambda)\delta_\alpha^\epsilon + (\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\lambda-\xi^\lambda)\delta_\alpha^\eta + (\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta)\delta_\alpha^\lambda]\} =$$

$$= - \frac{1}{12} \frac{\partial}{\partial\xi^\alpha} [\sum_{i=1}^{3} \lambda_i(\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta)(\xi_i^\lambda-\xi^\lambda)] ,$$

we finally obtain

$$\vec{a}_{h\alpha}(\xi) = \vec{a}_\alpha(\xi) + C_\alpha^{\epsilon\eta\lambda}(\xi)D_{\epsilon\eta\lambda}^k(\xi)\vec{a}_k(\xi) + 0(h^3)\vec{c}_\alpha \qquad (3.1.11)$$

where we have used the GAUSS-WEINGARTEN relations (see GREEN-ZERNA (1968)), and where we have set :

$$C_\alpha^{\epsilon\eta\lambda}(\xi) = - \frac{1}{12} \frac{\partial}{\partial\xi^\alpha} [\sum_{i=1}^{3} \lambda_i(\xi_i^\epsilon-\xi^\epsilon)(\xi_i^\eta-\xi^\eta)(\xi_i^\lambda-\xi^\lambda)] = 0(h^2) , \qquad (3.1.12)$$

$$D_{\epsilon\eta\lambda}^k(\xi) = \Gamma_{\epsilon\eta,\lambda}^k(\xi) + \Gamma_{\epsilon\eta}^j(\xi)\Gamma_{j\lambda}^k(\xi) ,$$

(with the conventions : $\Gamma_{\alpha\beta}^3 = b_{\alpha\beta}$, $\Gamma_{3\alpha}^\beta = -b_\alpha^\beta$, $\Gamma_{3\alpha}^3 = 0$).

Then, we derive :

$$a_{h\alpha\beta}(\xi) = a_{\alpha\beta}(\xi) + D_{\epsilon\eta\lambda}^\mu(\xi).[C_\alpha^{\epsilon\eta\lambda}(\xi)a_{\beta\mu}(\xi) + C_\beta^{\epsilon\eta\lambda}(\xi)a_{\alpha\mu}(\xi)] + 0(h^3)c_{\alpha\beta}$$

and, by using : $a_h = \frac{1}{2} e^{\alpha\lambda}e^{\beta\mu}a_{h\alpha\beta}a_{h\lambda\mu}$, we get :

$$a_h(\xi) = a(\xi)[1 + 2C_\mu^{\epsilon\eta\lambda}(\xi)D_{\epsilon\eta\lambda}^\mu(\xi)] + 0(h^3)c ,$$

and, in particular, we have :

$$\sqrt{\frac{a_h(\xi)}{a(\xi)}} = 1 + C_\mu^{\epsilon\eta\lambda}(\xi)D_{\epsilon\eta\lambda}^\mu(\xi) + 0(h^3)c . \qquad (3.1.13)$$

Next, by using the definition $\vec{a}_{h3} = \vec{a}_h^3 = (\vec{a}_{h1}\times\vec{a}_{h2})/\sqrt{a_h}$, we find :

$$\vec{a}_{h3}(\xi) = \vec{a}^3(\xi) - C_\alpha^{\epsilon\eta\lambda}(\xi)D^3_{\epsilon\eta\lambda}(\xi)\vec{a}^\alpha(\xi) + 0(h^3)\vec{c}_3 . \tag{3.1.14}$$

From (3.1.11) and (3.1.14), we obtain :

$$\left.\begin{aligned} d_{h\kappa}^{\ \ell}(\xi) &= \delta_\kappa^\ell + C_\alpha^{\epsilon\eta\lambda}(\xi)D^\ell_{\epsilon\eta\lambda}(\xi) + 0(h^3)c_\kappa^\ell , \\ d_{h3}^{\ \mu}(\xi) &= - C_\alpha^{\epsilon\eta\lambda}(\xi)D^3_{\epsilon\eta\lambda}(\xi)a^{\alpha\mu}(\xi) + 0(h^3)c_3^\mu , \\ d_{h3}^{\ 3}(\xi) &= 1 + 0(h^3)c_3^3 . \end{aligned}\right\} \tag{3.1.15}$$

Now, by using the relations $a_h^{\lambda\mu} = \dfrac{1}{a_h} e^{\lambda\alpha}e^{\mu\beta}a_{h\alpha\beta}$, $e_{\alpha\beta}e^{\lambda\mu} = \delta_\alpha^\lambda\delta_\beta^\mu - \delta_\beta^\lambda\delta_\alpha^\mu$,

$$a_h^{\lambda\mu}(\xi) = a^{\lambda\mu}(\xi) - D^\beta_{\epsilon\eta\alpha}(\xi)C_\kappa^{\epsilon\eta\alpha}(\xi)(\delta_\beta^\lambda a^{\kappa\mu}(\xi)+\delta_\beta^\mu a^{\kappa\lambda}(\xi)) + 0(h^3)c^{\lambda\mu} , \tag{3.1.16}$$

and, thus we derive (with $d^{h\lambda}_{\ \ \kappa}(\xi) = a_h^{\lambda\mu}(\xi)(\vec{a}_{h\mu}(\xi).\vec{a}_\kappa(\xi))$) :

$$d^{h\lambda}_{\ \ \kappa}(\xi) = \delta_\kappa^\lambda + 0(h^2)c_\kappa^\lambda . \tag{3.1.17}$$

Also, we have, directly from (3.1.14) :

$$d^{h3}_{\ \ k}(\xi) = \delta_k^3 + 0(h^2)c_k^3 , \tag{3.1.18}$$

and, from (3.1.11) and (3.1.16) :

$$d^{h\lambda}_{\ \ 3}(\xi) = 0(h^2)c_3^\lambda . \tag{3.1.19}$$

Moreover, from (3.1.11), we have :

$$\vec{a}_{h\alpha,\beta}(\xi) = \vec{a}_{\alpha,\beta}(\xi) + C^{\epsilon\eta\lambda}_{\alpha,\beta}(\xi)D^k_{\epsilon\eta\lambda}(\xi)\vec{a}_k(\xi) + 0(h^2)\vec{c}_{\alpha\beta} ;$$

therefore, we obtain :

$$\left.\begin{aligned}
\Gamma^{\alpha}_{h\beta\gamma}(\xi) &= a^{\alpha\nu}_{h}(\vec{a}_{h\nu}(\xi).\vec{a}_{h\beta,\gamma}(\xi)) \\
&= \Gamma^{\alpha}_{\beta\gamma}(\xi) + C^{\epsilon\eta\lambda}_{\beta,\gamma}(\xi)D^{\alpha}_{\epsilon\eta\lambda}(\xi) + 0(h^2)c^{\alpha}_{\beta\gamma}\ , \\
b_{h\alpha\beta}(\xi) &= \vec{a}_{h3}(\xi).\vec{a}_{h\alpha,\beta}(\xi) \\
&= b_{\alpha\beta}(\xi) + C^{\epsilon\eta\lambda}_{\alpha,\beta}(\xi)D^{3}_{\epsilon\eta\lambda}(\xi) + 0(h^2)c_{\alpha\beta}\ , \\
b^{\alpha}_{h\beta}(\xi) &= a^{\alpha\nu}_{h}(\xi)b_{h\nu\beta}(\xi) \\
&= b^{\alpha}_{\beta}(\xi) + a^{\alpha\nu}(\xi)C^{\epsilon\eta\lambda}_{\nu,\beta}(\xi)D^{3}_{\epsilon\eta\lambda}(\xi) + 0(h^2)c^{\alpha}_{\beta}\ , \\
b^{\alpha}_{h\beta}|_{\nu}(\xi) &= b^{\alpha}_{h\beta,\nu}(\xi) + \Gamma^{\alpha}_{h\nu\lambda}(\xi)b^{\lambda}_{h\beta}(\xi) - \Gamma^{\lambda}_{h\beta\nu}b^{\alpha}_{h\lambda}(\xi) \\
&= b^{\alpha}_{\beta}|_{\nu}(\xi) + a^{\alpha\kappa}(\xi)C^{\epsilon\eta\lambda}_{\kappa,\beta\nu}(\xi)D^{3}_{\epsilon\eta\lambda}(\xi) + 0(h)c^{\alpha}_{\beta\nu}\ .
\end{aligned}\right\} \quad (3.1.20)$$

Step 2 : Finite expansions of $\tilde{v}_{hk}(\xi)$, $\tilde{v}_{hk,\nu}(\xi)$, $1\le k\le 3$

Firstly, it is sufficient to recall the finite expansions (3.1.7)

$$\left.\begin{aligned}
\tilde{v}_{h\lambda}(\xi) &= v_{h\lambda}(\xi) + 0(h^2)\sum_{i=1}^{3}\{c^{j}_{\lambda}v_{hj}(\Sigma_i)\}\ , \\
\tilde{v}_{h\lambda,\nu}(\xi) &= v_{h\lambda,\nu}(\xi) + 0(h)\sum_{i=1}^{3}\{c^{j}_{\lambda\nu}v_{hj}(\Sigma_i)\}\ .
\end{aligned}\right\} \quad (3.1.21)$$

Secondly, we derive from (3.1.8)

$$\left.\begin{aligned}
\tilde{v}_{h3}(\xi)|_{K_j} &= v_{h3}(\xi)|_{K_j} + 0(h^2)\sum_{i=1}^{3}\{c^{k}_{j}v_{hk}(\Sigma_i) + c^{3\mu}_{j}v_{h3,\mu}(\Sigma_i) + c_j\omega^{3}_{h}(\Sigma_i)\}\ , \\
\tilde{v}_{h3,\nu}(\xi)|_{K_j} &= v_{h3,\nu}(\xi)|_{K_j} + 0(h)\sum_{i=1}^{3}\{c^{k}_{j}v_{hk}(\Sigma_i) + c^{3\mu}_{j}v_{h3,\mu}(\Sigma_i) + c_j\omega^{3}_{h}(\Sigma_i)\}\ .
\end{aligned}\right\} (3.1.22)$$

Step 3 : Finite expansion of $\tilde{v}_{h3,\alpha\beta}(\xi)$

By substituting (3.1.13) (3.1.15) (3.1.17) (3.1.19) into (3.1.3), we get for any $\xi \in K_j$ (subtriangle of K), $1\le j\le 3$:

$$\begin{aligned}
\tilde{v}_{h3,\alpha\beta}(\xi)|_{K_j} = v_{h3,\alpha\beta}(\xi)|_{K_j} &- \{\sum_{i=1}^{3}(p^{o}_{j,i})_{,\alpha\beta}C^{\epsilon\eta\lambda}_{\mu}(\Sigma_i) + \\
&+ \sum_{i=1}^{3}[(\xi^{\nu}_{i+1}-\xi^{\nu}_{i})(p^{1}_{j,i,i+1})_{,\alpha\beta} + (\xi^{\nu}_{i-1}-\xi^{\nu}_{i})(p^{1}_{j,i,i-1})_{,\alpha\beta}]C^{\epsilon\eta\lambda}_{\mu,\nu}(\Sigma_i)\} \times \\
&\qquad \times D^{3}_{\epsilon\eta\lambda}(\xi)a^{\mu\kappa}(\xi)v_{h\kappa}(\xi) + \\
&+ 0(h)\sum_{i=1}^{3}\{c^{k}_{j\alpha\beta}v_{hk}(\Sigma_i) + c^{k\lambda}_{j\alpha\beta}v_{hk,\lambda}(\Sigma_i) + c^{3\lambda\mu}_{j\alpha\beta}v_{h3,\lambda\mu}(\Sigma_i) + c_{j\alpha\beta}\omega^{3}_{h}(\Sigma_i)\}\ ,
\end{aligned}$$

Moreover, by noticing that we have :

$$\left.\begin{aligned}\sum_{k=1}^{3}\lambda_k(\xi)(\xi_k^\epsilon-\xi^\epsilon)(\xi_k^\eta-\xi^\eta)(\xi_k^\lambda-\xi^\lambda) &= \sum_{k=1}^{3}\lambda_k(\xi)\xi_k^\epsilon\xi_k^\eta\xi_k^\lambda - \xi^\epsilon\sum_{k=1}^{3}\lambda_k(\xi)\xi_k^\eta\xi_k^\lambda - \\ &- \xi^\eta\sum_{k=1}^{3}\lambda_k(\xi)\xi_k^\epsilon\xi_k^\lambda - \xi^\lambda\sum_{k=1}^{3}\lambda_k(\xi)\xi_k^\epsilon\xi_k^\eta + 2\xi^\epsilon\xi^\eta\xi^\lambda ,\end{aligned}\right\}$$

and, thus from (3.1.12) :

$$\begin{aligned}-12C_\mu^{\epsilon\eta\lambda}(\xi) &= [\sum_{k=1}^{3}\frac{\partial\lambda_k}{\partial\xi^\mu}\xi_k^\epsilon\xi_k^\eta\xi_k^\lambda] - [\sum_{k=1}^{3}\lambda_k(\xi)(\delta_\mu^\epsilon\xi_k^\eta\xi_k^\lambda+\delta_\mu^\eta\xi_k^\epsilon\xi_k^\lambda+\delta_\mu^\lambda\xi_k^\epsilon\xi_k^\eta) + \\ &+ \xi^\epsilon\sum_{k=1}^{3}\frac{\partial\lambda_k}{\partial\xi^\mu}\xi_k^\eta\xi_k^\lambda + \xi^\eta\sum_{k=1}^{3}\frac{\partial\lambda_k}{\partial\xi^\mu}\xi_k^\epsilon\xi_k^\lambda + \xi^\lambda\sum_{k=1}^{3}\frac{\partial\lambda_k}{\partial\xi^\mu}\xi_k^\epsilon\xi_k^\eta] + \\ &+ 2[\delta_\mu^\epsilon\xi^\eta\xi^\lambda + \delta_\mu^\eta\xi^\epsilon\xi^\lambda + \delta_\mu^\lambda\xi^\epsilon\xi^\eta] ,\end{aligned}$$

in such a manner that : $C_\mu^{\epsilon\eta\lambda}(\xi) \in P_2(K)$, the function $C_\mu^{\epsilon\eta\lambda}(\xi)$ is then invariant by the HCTr-interpolation, and we obtain :

$$\begin{aligned}C_{\mu,\alpha\beta}^{\epsilon\eta\lambda}(\xi) &= \sum_{i=1}^{3}(p_{j,i}^{o})_{,\alpha\beta}C_\mu^{\epsilon\eta\lambda}(\Sigma_i) + \\ &+ \sum_{i=1}^{3}[(\xi_{i+1}^\nu-\xi^\nu)(p_{j,i,i+1}^{1})_{,\alpha\beta} + (\xi_{i-1}^\nu-\xi_i^\nu)(p_{j,i,i-1}^{1})_{,\alpha\beta}]C_{\mu,\nu}^{\epsilon\eta\lambda}(\Sigma_i) .\end{aligned}$$

So, we deduce :

$$\begin{aligned}\tilde{v}_{h3,\alpha\beta}(\xi)|_{K_j} &= v_{h3,\alpha\beta}(\xi)|_{K_j} - C_{\mu,\alpha\beta}^{\epsilon\eta\lambda}(\xi)D_{\epsilon\eta\lambda}^{3}(\xi)a^{\mu\nu}(\xi)v_{h\nu}(\xi) + \\ &+ O(h)\sum_{i=1}^{3}\{c_{j\alpha\beta}^{k}v_{hk}(\Sigma_i) + c_{j\alpha\beta}^{k\lambda}v_{hk,\lambda}(\Sigma_i) + c_{j\alpha\beta}^{3\lambda\mu}v_{h3,\lambda\mu}(\Sigma_i) + c_{j\alpha\beta}\omega_h^3(\Sigma_i)\}\end{aligned} \tag{3.1.23}$$

Step 4 : Finite expansion of $\tilde{\rho}_{h\beta}^{\alpha}(\vec{\vec{v}}_h)$

By combining (3.1.16) (3.1.20) to (3.1.23) in (3.1.10), we obtain for any $\xi \in K_j$:

$$\begin{aligned}\tilde{\rho}_{h\beta}^{\alpha}(\vec{\vec{v}}_h)|_{K_j} &= \bar{\rho}_{\beta}^{\alpha}(\vec{v}_h)|_{K_j} + \\ &+ O(h)\sum_{i=1}^{3}\{c_{\beta j}^{\alpha k}v_{hk}(\Sigma_i) + c_{\beta j}^{\alpha k\lambda}v_{hk,\lambda}(\Sigma_i) + c_{\beta j}^{\alpha\lambda\mu}v_{h3,\lambda\mu}(\Sigma_i) + c_{\beta j}^{\alpha}\omega_h^3(\Sigma_i)\} ,\end{aligned} \tag{3.1.24}$$

which allows us to establish the estimate (3.1.9).

□

Thirdly, we obtain in a similar way :

Theorem 3.1.3 : There exists a constant C, *independent of* h, *such that for any* $(\vec{\vec{v}}_h,\tilde{\omega}_h^3) \in \vec{\vec{W}}_h$ *and* $(\vec{v}_h,\omega_h^3) \in \vec{W}_h$ *in correspondence through the bijection* F_h, *we have :*

$$|\tilde{\omega}_h^3 - \omega_h^3|_{0,K} \leq Ch^2(\|v_{h1}\|_{0,K}^2 + \|v_{h2}\|_{0,K}^2 + \|v_{h3}\|_{1,K}^2 + \|\omega_h^3\|_{0,K}^2)^{1/2} . \tag{3.1.25}$$

Proof :

From the definition of the space $\tilde{\tilde{X}}_{h1}$, we have for any $\xi \in K$:

$$\tilde{\omega}_h^3(\xi) = \sum_{i=1}^{3} \lambda_i \tilde{\omega}_h^3(\Sigma_i) ,$$

and, by using (2.2.9) (3.1.15) (3.1.18), we get :

$$\tilde{\omega}_h^3(\xi) = \omega_h^3(\xi) + 0(h^2) \sum_{i=1}^{3} \{c^\lambda v_{h\lambda}(\Sigma_i) + c^\nu v_{h3,\nu}(\Sigma_i) + c_3\omega_h^3(\Sigma_i)\} , \tag{3.1.26}$$

which lead to the inequality (3.1.25).

□

Consequently, from the estimates (3.1.1) (3.1.9) (3.1.13) and (3.1.25), we deduce :

Theorem 3.1.4 : For any regular triangulation $\mathcal{T}_h$ *of* $\bar{\Omega}$, *there exists a constant* C, *independent of* h, *such that for any* $(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3) \in \vec{W}_h$, *we have :*

$$\left.\begin{aligned} &|a(\vec{v}_h,\vec{w}_h) + k(\omega_h^3,\theta_h^3)_{L^2(\Omega)} - A_h[(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3)]| \leq \\ &\qquad \leq Ch(\|\vec{v}_h\|_{\vec{V}}^2+\|\omega_h^3\|_{0,\Omega}^2)^{1/2} (\|\vec{w}_h\|_{\vec{V}}^2+\|\theta_h^3\|_{0,\Omega}^2)^{1/2} . \end{aligned}\right\} \tag{3.1.27}$$

Proof :

Firstly, the estimates (3.1.1) (3.1.9) (3.1.13) implies that

$$\begin{aligned} &\sum_{K\in\mathcal{T}_h} |a(\vec{v}_h,\vec{w}_h)_{|K} - \tilde{a}_{Kh}(\vec{\vec{v}}_h,\vec{\vec{w}}_h)| \leq \\ &\qquad \leq Ch(\|\vec{v}_h\|_{\vec{V}}^2+\|\omega_h^3\|_{0,\Omega}^2)^{1/2} (\|\vec{w}_h\|_{\vec{V}}^2+\|\theta_h^3\|_{0,\Omega}^2)^{1/2} , \end{aligned} \tag{3.1.28}$$

for any $(\vec{\vec{v}}_h,\tilde{\omega}_h^3),(\vec{\vec{w}}_h,\tilde{\theta}_h^3) \in \vec{\vec{W}}_h$, and $(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3) \in \vec{W}_h$, respectively in correspondence through the bijection F_h ; similarly, by using (II, (4.2.1)) and (3.1.25), we derive that :

$$\left.\begin{aligned}\sum_{K\in\mathscr{T}_h} &\left|(\omega_h^3,\theta_h^3)_{L^2(K)} - (\tilde{\omega}_h^3,\tilde{\theta}_h^3)_{L^2(K)}\right| \le \\ &\le Ch^2(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega_h^3\|^2_{0,\Omega})^{1/2}\ (\|\vec{w}_h\|^2_{\vec{V}}+\|\theta_h^3\|^2_{0,\Omega})^{1/2}\ .\end{aligned}\right\} \quad (3.1.29)$$

Then, by using the relation :

$$a(\vec{v}_h,\vec{w}_h) + k(\omega_h^3,\theta_h^3)_{L^2(\Omega)} - A_h[(\vec{v}_h,\omega_h^3),(\vec{w}_h,\theta_h^3)] =$$

$$= \sum_{K\in\mathscr{T}_h} \{[a(\vec{v}_h,\vec{w}_h)_{|K} - \tilde{a}_{Kh}(\vec{\vec{v}}_h,\vec{\vec{w}}_h)] + k[(\omega_h^3,\theta_h^3)_{L^2(K)} - (\tilde{\omega}_h^3,\tilde{\theta}_h^3)_{L^2(K)}]\}\ ,$$

we derive, from (3.1.28) and (3.1.29), the inequality (3.1.27), where the constant C may be chosen independent of k as soon as k is taken in a compact set $[k_0,k_1]$.

□

3.2. Estimate of $|f(\vec{v}_h) - G_h[(\vec{v}_h,\omega_h^3)]|$

Next, we can establish :

Theorem 3.2.1 : For any regular triangulation $\mathscr{T}_h$ *of* $\bar{\Omega}$*, and for any* $\vec{p} \in (L^2(\Omega))^3$ *, there exists a constant* C*, independent of* h*, such that for any* $(\vec{v}_h,\omega_h^3) \in \vec{W}_h$ *, we have :*

$$|f(\vec{v}_h) - G_h[(\vec{v}_h,\omega_h^3)]| \le Ch^2(\|\vec{v}_h\|^2_{\vec{V}}+\|\omega_h^3\|_{0,\Omega})^{1/2}\ \|\vec{p}\|_{0,\Omega}\ . \quad (3.2.1)$$

Proof :

This proof is similar to that one of (II, Theorem 3.4.1) and uses the decomposition :

$$f(\vec{v}_h) - G_h[(\vec{v}_h,\omega_h^3)] = \sum_{K\in\mathscr{T}_h} [f(\vec{v})|_K - \tilde{f}_{Kh}(\vec{\vec{v}}_h)]\ ,$$

for any $(\vec{\vec{v}}_h,\tilde{\omega}_h^3) \in \vec{\vec{W}}_h$ and $(\vec{v}_h,\omega_h^3) \in \vec{W}_h$ in correspondence through the bijection F_h. Thus, by using (II, (3.4.2)), (3.1.4), (3.1.7), (3.1.8)), we derive :

$$\left.\begin{aligned}&|f(\vec{v}_h)_{|K} - \tilde{f}_{Kh}(\vec{\vec{v}}_h)| \le \\ &\qquad \le Ch^2(\|v_{h1}\|^2_{0,K} + \|v_{h2}\|^2_{0,K} + \|v_{h3}\|^2_{1,K} + \|\omega_h^3\|^2_{0,K})^{1/2}\ \|\vec{p}\|_{0,\Omega}\ ,\end{aligned}\right\} \quad (3.2.2)$$

and estimate (3.2.1) follows.

□

3.3. Existence and uniqueness of $(\vec{\vec{u}}_h, \tilde{\eta}_h^3)$ *; convergence*

Henceforth, we complete the analysis of the method for arbitrary thin shells :

Theorem 3.3.1 : Let $\mathcal{T}_h$ *be a regular family of triangulations of the domain* $\bar{\Omega}$. *Then, if the solution* $\vec{u} \in \vec{V}$ *belongs to the space* $(H^2(\Omega))^2 \times H^3(\Omega)$ *, if the loads* $\vec{p} \in (L^2(\Omega))^3$ *, there exists constants* $0<k_0<k_1$ *,* $h_1>0$ *, such that for any* k *,* $k_0 \le k \le k_1$ *, for any* $h<h_1$ *,*

(i) the discrete problem 2.2.2 (respectively problem 2.2.1) has one and only one solution $(\vec{u}_h^*, \eta_h^3) \in \vec{W}_h$ *(resp.* $(\vec{\vec{u}}_h, \tilde{\eta}_h^3) \in \vec{\vec{W}}_h$*) ;*

(ii) there exists a positive constant C *, independent of* h *, such that :*

$$(\|\vec{u}-\vec{u}_h^*\|_{\vec{V}}^2 + \|\eta_h^3\|_{0,\Omega}^2) \le Ch\{(\|u_1\|_{2,\Omega}^2 + \|u_2\|_{2,\Omega}^2 + \|u_3\|_{3,\Omega}^2)^{1/2} + \|\vec{p}\|_{0,\Omega}\} \ . \qquad (3.3.1)$$

Proof :

This proof is similar to that one of (II, Theorem 5.3.1)

From the estimate (3.1.27), and from the uniform $\vec{V}$-ellipticity of the bilinear form $a(.,.)$, we derive that :

$$\left.\begin{aligned} & A_h[(\vec{v}_h,\omega_h^3),(\vec{v}_h,\omega_h^3)] \ge \\ & \qquad \ge [\min(\alpha,k) - Ch] \ (\|\vec{v}_h\|_{\vec{V}}^2 + \|\omega_h^3\|_{0,\Omega}^2) \ , \quad \forall (\vec{v}_h,\omega_h^3) \in \vec{W}_h \ . \end{aligned}\right\} \qquad (3.3.2)$$

On the one hand, when $\min(\alpha,k_0) \ge Ch_1$, the inequality (3.3.2) establishes the existence and uniqueness of the solution $(\vec{u}_h^*, \eta_h^3) \in \vec{W}_h$ for the discrete problem 2.2.2, and thus, the existence and uniqueness of $(\vec{\vec{u}}_h, \tilde{\eta}_h^3) \in \vec{\vec{W}}_h$ for the discrete problem 2.2.1, by using the bijection F_h defined in Theorem 2.2.1, i.e. $(\vec{\vec{u}}_h, \tilde{\eta}_h^3) = F_h^{-1}(\vec{u}_h^*, \eta_h^3)$. On the other hand, when $\min(\alpha,k_0) \ge Ch_1$, the inequality (3.3.2) allows us to find an abstract error estimate of the type (II, 4.1.2). Therefore, by using the interpolation error estimate (II, (4.5.3)) and the consistency error estimates (3.1.27) and (3.2.1), we obtain (3.3.1).

□

Remark 3.3.1 : It is possible to take into account the effect of numerical integration in a similar manner as in Part I : this would result in a similar estimate as (3.3.1).

□

4 - *REFERENCES*

BATHE, K.J. (1982) : Finite Element Procedures in Engineering Analysis. Prentice-Hall.

BERNADOU, M. (1980) : Convergence of conforming finite element methods for general shell problems. Int. J. Engng. Sc., 18, 249-276.

BERNADOU, M. ; BOISSERIE, J.M. (1982) : The Finite Element Method in Thin Shell Theory : Application to Arch Dam Simulations. Birkhaüser, Boston.

BERNADOU, M. ; CIARLET, P.G. (1976) : Sur l'ellipticité du modèle linéaire de W.T. Koiter "Computing Methods in Applied Sciences and Engineering". Lecture Notes in Economic and Mathematical Systems, Vol. 134, 89-136, Springer-Verlag, Berlin.

BERNADOU, M. ; DUCATEL, Y. ; TROUVÉ, P. (Submitted to Comp. Mech.) : Approximation of general shell problems by flat plate elements, Part 1.

BERNADOU, M. ; TROUVÉ, P. (for submission to Comp. Mech.) : Approximation of general shell problems by flat plate elements, Part 2 : Addition of a drilling degree of freedom.

CIARLET, P.G. (1976) : Conforming finite element method for the shell problems, in the Mathematics of Finite Elements and Applications II, J.R. Whiteman Ed., Academic Press, London, 105-123.

CIARLET, P.G. (1978) : The Finite Element Method for Elliptic Problems. North-Holland.

DUPUIS, G. ; GOEL, J.J. (1970) : A curved finite element for thin elastic shells, Int. J. Solids and Structures, 6, 1413-1428.

GREEN, A.E. ; ZERNA, W. (1968) : Theoretical Elasticity. Oxford University Press, 2nd Ed.

IDELHSON, S. (1981) : On the use of deep, shallow or flat shell finite elements for the analysis of thin shell structures. Comp. Meth. Appl. Mech. Engng., 26, 321-330.

KIKUCHI, F. (1984) : Error analysis of flat plate element approximation of circular cylindrical shells. Theoretical and Applied Mechanics, 32, 469-484.

KOITER, W.T. (1970) : On the foundations of the linear theory of thin elastic shells. Proc. Kon. Ned. Akad. Wetensch., B73, 169-195.

MORLEY, L.S.D. (1983) : Study of trial functions in shell triangular finite elements of quadratic parametric representation. Comp. Meth. in Appl. Mech. and Engng., 38, 203-236.

MORLEY, L.S.D. (1984) : Finite element criteria for some shells. Int. J. Num. Meth. Engng., 20, 1711-1728.

ZIENKIEWICZ, O.C. (1977) : The Finite Element Method in Engineering Science, 3rd Ed., Mc Graw-Hill.

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